Math 347: Lecture 4 - Worksheet

September 5, 2018

1) What is the relation between the two following statements?

$$(\forall x \in A)(\exists y \in B)P(x, y) \quad \text{and} \quad (\exists y \in B)(\forall x \in A)P(x, y).$$

Find examples of A and B that justify your claim.

2) Is the following statement true or false?

$$(\exists a, b \in \mathbb{R})(\forall x \in \mathbb{R})(ax^2 + bx \neq a).$$

How does this relate to Example 2.10 from the book?

3) A function $f : \mathbb{R} \to \mathbb{R}$ is *bounded* if it satisfies the following

$$(\exists M \in \mathbb{R})(\forall x \in \mathbb{R})(|f(x)| \leq M).$$

We say that f is unbounded if the above statement is false. Prove that if f is unbounded then

$$(\forall n \in \mathbb{N})(\exists x_n \in \mathbb{R})(|f(x_n)| > n).$$

4) Show that the following statement is false: "If a and b are integers, then there are integers m and n such that $a = m + n$ and $b = m - n$." What can be added to the hypothesis of the statement to make it true?

5) Let $P(x)$ be the statement "x is odd", and $Q(x)$ the statement "$x^2 - 1$ is divisible by 8". Determine whether the following are true:

   (i) $(\forall x \in \mathbb{Z})[P(x) \Rightarrow Q(x)]$,

   (ii) $(\forall x \in \mathbb{Z})[Q(x) \Rightarrow P(x)]$.

6) For $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ show that (i) and (ii) have different meanings:

   (i) $(\forall \epsilon > 0)(\exists \delta > 0)[(|x - a| < \delta) \Rightarrow (|f(x) - f(a)| < \epsilon)]$

   (ii) $(\exists \delta > 0)(\forall \epsilon > 0)[(|x - a| < \delta) \Rightarrow (|f(x) - f(a)| < \epsilon)]$

   Can you come up with examples of a and f that satisfy or do not satisfy the above?

7) Prove the following identities about sets.

   (i) $(A \cup B)^c = A^c \cap B^c$ (de Morgan's law);

   (ii) $A \cap [(A \cap B)^c] = A - B$;

   (iii) $A \cap [(A \cap B^c)^c] = A \cap B$;

   (iv) $(A \cup B) \cap A^c = B - A$.